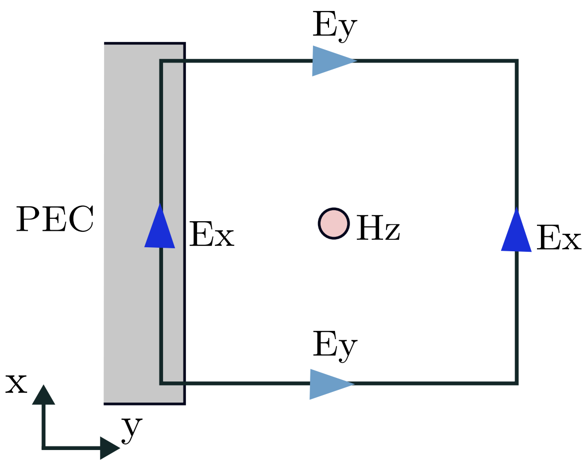
1 PEC Edge Singularity Treatment



H_z and E_y vary as $1/\sqrt{r}$.
For the Hz, update, instead of assuming the E_x fields are constant over the cell width as in the normal FDTD updates,use the $1/\sqrt{r}$ model.
The H_z field at a distance r from the PEC edge is,

$$H_z(r) = \frac{H_{z0}}{\sqrt{r}}$$

The constant H_{z0} can be found since the field at $\frac{\Delta y}{2}$ is the $H_z(i, j, k)$ component in the Yee grid,

$$H_z\Big(r = \frac{\Delta y}{2}\Big) = H_{z0}\Big(\frac{\Delta y}{2}\Big)^{-\frac{1}{2}} = H_z(i, j, k)$$

$$H_{z0} = H_z(i, j, k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}}$$

So the H_z at any point r is,

$$H_z(r) = H_z(i, j, k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} r^{-\frac{1}{2}}$$

By similar logic, since E_y is at the same point along r as H_z,

$$E_y(r) = E_y(i, j, k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} r^{-\frac{1}{2}}$$

Plugging this in to the update equation for H_z,

$$\mu_0 \int_S \frac{\partial H_z}{\partial t} dS = \Big[E_x(i, j+1, k) - E_x(i, j, k)\Big]\Delta x - \int_{r=0}^{r=\Delta y} \Big[E_y(i+1, k) - E_y(i, k)\Big] dr$$

The surface integral over H_z (the left side term) is,

$$\mu_0 \int_{x=0}^{x=\Delta x} \int_{r=0}^{r=\Delta y} \frac{\partial}{\partial t} H_z(i, j, k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} r^{-\frac{1}{2}} dx dr$$

$$= \Delta x \mu_0 \frac{\partial H_z(i,j,k)}{\partial t} \Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} \Big[2\ r^{\frac{1}{2}}\Big]_{r=0}^{r=\Delta y}$$

$$= 2 \Delta x \mu_0 \Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} \frac{\partial H_z(i,j,k)}{\partial t} (\Delta y)^{\frac{1}{2}}$$

$$= 2\sqrt{\frac{1}{2}} \Delta x \Delta y \mu_0 \frac{\partial H_z(i,j,k)}{\partial t}$$

The E_y terms are similar,

$$-\int_{r=0}^{r=\Delta y} \Big[E_y(i+1,j,k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} r^{-\frac{1}{2}} - E_y(i,j,k)\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} r^{-\frac{1}{2}}\Big] dr$$

$$= -\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} \Big[E_y(i+1,j,k) - E_y(i,j,k)\Big] \int_{r=0}^{r=\Delta y} r^{-\frac{1}{2}} dr$$

$$= -\Big(\frac{\Delta y}{2}\Big)^{\frac{1}{2}} \Big[E_y(i+1,j,k) - E_y(i,j,k)\Big] (2 \Delta y^{\frac{1}{2}})$$

$$= -2\sqrt{\frac{1}{2}} \Delta y \Big[E_y(i+1,j,k) - E_y(i,j,k)\Big]$$

Using variables for the correction terms,

$$CF_{Hz} = CF_{Ey} = 2\sqrt{\frac{1}{2}} =$$

The full update equation is then,

$$CF_{Hz} \Delta x \Delta y \mu_0 \frac{\partial H_z(i,j,k)}{\partial t} =$$

$$\Big[E_x(i,j+1,k) - E_x(i,j,k)\Big] \Delta x - CF_{Ey} \Delta y \Big[E_y(i+1,j,k) - E_y(i,j,k)\Big]$$

Applying central differencing to the H_z term,

$$CF_{Hz} \Delta x \Delta y \mu_0 \frac{H_z^{n+1}(i,j,k) - H_z^n(i,j,k)}{\Delta t} =$$

$$\Big[E_x(i,j+1,k) - E_x(i,j,k)\Big] \Delta x - CF_{Ey} \Delta y \Big[E_y(i+1,j,k) - E_y(i,j,k)\Big]$$

Solving for H_z^{n+1},

$$H_z^{n+1}(i,j,k) = H_z^n(i,j,k) +$$

$$\Big[E_x(i,j+1,k) - E_x(i,j,k)\Big] \frac{\Delta t}{\mu_0 \Delta y} \frac{1}{CF_{Hz}}$$

$$-\Big[E_y(i+1,j,k) - E_y(i,j,k)\Big] \frac{\Delta t}{\mu_0 \Delta x} \frac{CF_{Ey}}{CF_{Hz}}$$

The derivation for the E component updates is analogous. For E_z and a non-linear H_y component this is,

$$E_z^{n+1}(i,j,k) = E_z^n(i,j,k) +$$

$$\Big[H_y(i,j+1,k) - H_y(i,j,k)\Big]\frac{\Delta t}{\varepsilon_0 \Delta x}\frac{CF_{Hy}}{CF_{Ez}}$$
$$-\Big[H_x(i+1,j,k) - H_x(i,j,k)\Big]\frac{\Delta t}{\varepsilon_0 \Delta y}\frac{1}{CF_{Ez}}$$

If the fields have more complicated variations than simply $\frac{1}{\sqrt{r}}$, they can be found numerically using a very fine mesh. The correction terms are simply the number that $H_z \Delta z$ (for example) needs to be multiplied by to reach the true value of the integral using the actual non-linear field variation.

$$CF = \frac{\displaystyle\int_{r=0}^{r=\Delta} H_z(r)dr}{H_z(i,j,k)\Delta z}$$

If the field has a r^{-v} dependence instead of the $v = 1/2$ used above,

$$H_z(r) = H_z(i,j,k)\Big(\frac{\Delta z}{2}\Big)^v r^{-v}$$

$$CF = \frac{1}{\Delta z}\Big(\frac{\Delta z}{2}\Big)^v \int_{r=0}^{r=\Delta z} r^{-v} dr$$

$$CF = \frac{1}{\Delta z}\Big(\frac{\Delta z}{2}\Big)^v \frac{(\Delta z)^{1-v}}{(1-v)}$$

$$CF = \Big(\frac{1}{2}\Big)^v \frac{1}{(1-v)}$$

2 E_z Correction

The E_z component above and below the edge of the trace varies along y as,

$$E_z(a) = E_{z,0}\Big(\frac{\Delta z/2}{a^2 + (\Delta z/2)^2)}\Big)$$

where $a = 0$ is directly under the edge.
$E_{z,0}$ can be found by setting $a = 0$, which corresponds to the $E_z(i,j,k)$ component in the Yee grid.

$$E_{z,0} = E_z(a=0)\frac{\Delta z}{2} = E_z(i,j,k)\frac{\Delta z}{2}$$

The correction factor is then,

$$CF = \frac{\displaystyle\int_{a=-\Delta y/2}^{a=\Delta y/2} E_{z,0}\Big(\frac{\Delta z/2}{a^2 + (\Delta z/2)^2}\Big)da}{E_z(i,j,k)\Delta y}$$

$$CF = \frac{\Delta z}{2\Delta y}\int_{a=-\Delta y/2}^{a=\Delta y/2} \frac{\Delta z/2}{a^2 + (\Delta z/2)^2} da$$

$$CF = \left(\frac{\Delta z}{2}\right)^2 \frac{1}{\Delta y} \int_{a=-\Delta y/2}^{a=\Delta y/2} \frac{1}{a^2 + (\Delta z/2)^2} da$$

$$CF = \left(\frac{\Delta z}{2}\right)^2 \frac{1}{\Delta y} \left[\frac{\tan^{-1}\left(\frac{a}{\Delta z/2}\right)}{\Delta z/2}\right]_{a=-\Delta y/2}^{a=\Delta y/2}$$

$$CF = \left(\frac{\Delta z}{2}\right) \frac{1}{\Delta y} \left[\tan^{-1}\left(\frac{a}{\Delta z/2}\right)\right]_{a=-\Delta y/2}^{a=\Delta y/2}$$

$$CF = \left(\frac{\Delta z}{2}\right) \frac{1}{\Delta y} \left[\tan^{-1}\left(\frac{\Delta y/2}{\Delta z/2}\right) - \tan^{-1}\left(\frac{-\Delta y/2}{\Delta z/2}\right)\right]$$

$$CF = \left(\frac{\Delta z}{2}\right) \frac{1}{\Delta y} \left[2\tan^{-1}\left(\frac{\Delta y/2}{\Delta z/2}\right)\right]$$

$$CF = \frac{\Delta z}{\Delta y} \tan^{-1}\left(\frac{\Delta y}{\Delta z}\right)$$

H_y has the same correction factor along y under and above the trace edge.